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FRAGMENTATION OF METALS AT HIGH STRAINS: A MECHANISM OF FORMATION OF SPATIALLY-MODULATED VORTEX STRUCTURES

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Fragmentation of the structure of lattice disorientations at high plastic strains of metal crystals is studied. The medium is modelled by the geometrically nonlinear elastoplastic Cosserat continuum. The points of the continuum are identified with dislocation cells with a frozen crystal lattice.

Description of the Fragmentation Phenomenon. In the present paper, the mechanism of formation of a fragmented structure of lattice disorientations in plastically deformable metals is studied. In the experiments [1, 2], specimens were subjected to quasi-static "uniaxial" tension or compression (rolling). The specimen material was metal polycrystals with a usual grain size (tens of micrometers), FCC lattice, and relatively high stacking-fault energy. For high plastic strains and stresses, fragmentation of the grains of a polycrystal followed by rotation of the lattice was observed. This process occurred in a stabilized dislocation cellular structure; therefore, these fragments (Fig. 1) were called the cell blocks [2]. A similar pattern is shown in [1, Fig. 12]. The size of the dislocation cells was of the order of several tenths of a micrometer, and the size of the fragments was of the order of a micrometer; therefore, this process can be considered as the formation of large-scale blocks (fragments) in a granular (cellular) medium. Practically random rotations of the cell lattice were of the order of tenths of a degree, whereas the rotations of a fragment lattice were as large as several degrees or several tens of degrees.

It should be noted that this phenomenon is similar to the occurrence of spatially periodic vortex structures at high deformations of sand saturated with a viscous liquid (see [3, Fig. 7.19]).

Fragmentation Mechanism. Below, we explain the phenomenon in question using the following assumptions.

1. In a current stress configuration, the medium is a dense aggregate consisting of absolutely rigid particles, which can perform small translational and rotational motions. The prototype of these particles is dislocation cells; the assumption that the particles are rigid at the current moment means that the crystal lattice is frozen into these particles. In a smooth approximation, this medium is the Cosserat continuum. For description of the formation of periodic vortex structures, it is important that the model of this continuum naturally takes into account the rotations and contains a characteristic size.

2. The structural change in the stressed medium is due to instability of a local affine distortion of the medium. At the critical moment, the medium undergoing affine deformation cannot sustain the load, and nonaffine deformation modes become pronounced. This results in the occurrence of a spatially modulated structure of a twisted and bent lattice. In particular, a cellular structure with spins alternating in direction can be formed, which leads to the formation of fragments of lattice disorientations.

It is assumed that this mechanism works in a certain region of a stressed medium whose dimensions are much greater than the characteristic size of fragments and whose properties and fields are nearly homogeneous. Under different conditions, other fragmentation mechanisms can work at the earlier stages of loading.

The foregoing explains the mechanical (external) aspect of the phenomenon. Using the terms of evolution of dislocation and above-dislocation systems, we can say that the physical (internal) aspect of the phenomenon should follow this mechanism.

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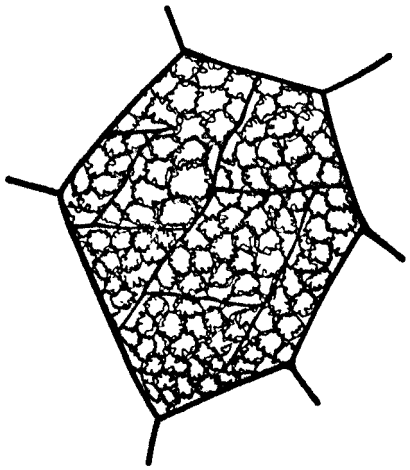

Fig. 1

Much research is devoted to structural transitions (bifurcations) in open nonlinear dissipative systems. In their classical works, Van der Waals, Ginzburg, and Landau revealed the fundamental role of the gradient of the variable of state in the neighborhood of the critical point. It is precisely the gradient term that withstands the forces which act just beyond the point of the local maximum of the strongly nonlinear dependence of the generalized force on the generalized displacement. Owing to the formation of structures, this term ensures the absorption of the free energy released. Aifantis [4] studied conceptually similar models of formation of structures in plastic metals. Aéro [5] considered an essentially nonlocal model with a finite number of critical points characteristic of the periodic potential of an atomic lattice.

The Cosserat continuum contains the gradient term. Let us study the instability mechanism. It was found experimentally that the cellular structure is stabilized before fragmentation. It follows that the hardening moduli decrease since the flow stress in coarse-grain metal polycrystals is uniquely related to the average size of dislocation cells by the Petch–Hall experimental relation. For reasonably small hardening moduli and high loads, the instability can be caused by local changes in the geometry [6], which is supported by experimental studies of metals [7]. In this case, the curve of nominal stress versus strain has a maximum. A decrease in the nominal stress (rather than actual stress) behind the maximum point should be regarded as a critical condition in the geometrically nonlinear formulation of elastoplastic problems [8, 9]. Moreover, this type of nonlinearity does not imply any jumps and allows us to confine our attention to the linearized equations of equilibrium in the neighborhood of the single maximum point.

Equations of the Model. Assuming that a medium in a current stressed configuration K has uniformly distributed properties and uniform lattice-orientation and displacement fields, which are in equilibrium with the applied stress, we obtain linearized dynamic equations for disturbances of these fields in the neighborhood of K. Trivial solutions of these equations correspond to the "basic", undisturbed state of the body. Of interest, however, are nontrivial, "disturbed" solutions and conditions for their existence. Stability of elastoplastic systems is usually studied with the use of the bifurcation or energy approach. Nevertheless, a particular case of the model formulated below corresponds to circulating systems, which can be analyzed adequately only by the dynamic method [10]. Therefore, we use this method in the present study.

We confine ourselves to the two-dimensional case. In the current configuration K, a Cartesian orthogonal system of material coordinates with the orthonormal basis $\boldsymbol{e}_\alpha$ is determined (hereafter, the Greek subscripts change by modulus 2 and the Latin subscripts change by modulus 3; summation is performed over repeated subscripts). Linearized equations of dynamics and state are written in this coordinate system. This approach, which is called the Lagrangian current approach, was used in [6–9]. As in these studies, the incremental-linear dynamic equations are written in terms of disturbance velocities.

The equations

$$\begin{gathered} \nabla \cdot \dot{\boldsymbol{\sigma}} - \nabla \boldsymbol{v}^{\mathrm{t}} \circ \nabla \boldsymbol{\sigma} = \rho \ddot{\boldsymbol{v}}, \\ \nabla \cdot \dot{\boldsymbol{\mu}} - \nabla \boldsymbol{v}^{\mathrm{t}} \circ \nabla \boldsymbol{\mu} + \boldsymbol{\mathcal{E}} \circ \dot{\boldsymbol{\sigma}} = J \ddot{\boldsymbol{\omega}} \end{gathered} \tag{1}$$

are obtained by material differentiation of the dynamic equations of the Cosserat continuum [11] with allowance for the distortion of the configuration caused by instantaneous motion of the material: $\dot{\nabla} = -(\nabla \boldsymbol{v}) \cdot \nabla$. In (1), $\boldsymbol{\sigma}$ and $\boldsymbol{\mu}$ are the stress and moment (couple) tensors, respectively, $\nabla \equiv \boldsymbol{e}_\alpha \partial_\alpha$, $\boldsymbol{v}$ is the velocity vector of the translational motion of the particle, $\boldsymbol{\omega} = \omega \boldsymbol{e}_3$ is the vector of angular velocity of the particle, which can be expressed in terms of the antisymmetric spin tensor $\boldsymbol{\Omega} \equiv -\boldsymbol{\mathcal{E}} \cdot \boldsymbol{\omega}$, ρ and J are the density of mass and moment of inertia of the particle, respectively, and $\boldsymbol{\mathcal{E}}$ is the Levi-Civita tensor; the circle denotes the convolution of the second-rank tensors $\boldsymbol{A} \circ \boldsymbol{B} \equiv A_{\alpha\beta} B_{\alpha\beta}$ in the basis $\boldsymbol{e}_\alpha \boldsymbol{e}_\beta$.

We determine the structure of the incremental-linear equations of state for an elastoplastic granular material. Below, we consider equally active bifurcations; therefore, we use the equations of state only for the branch of active loading [8, 9, 12]. The metal is assumed to be incompressible.

We consider three rigid particles A, B, and C that contact each other in pairs. The particles have the unit normal ($\boldsymbol{n}_k$) and tangent ($\boldsymbol{b}_k$) vectors at the contact points $k = 1, 2, 3$. The dyads $\boldsymbol{n}_k \boldsymbol{b}_k \equiv \boldsymbol{M}_k$ (no summation over k), where $\operatorname{tr} \boldsymbol{M}_k = 0$, form the basis in the space of nonsymmetric deviators above the plane. The relative velocity of the kth pair of particles (say, A and B) at the contact point is $\boldsymbol{v}_k = \boldsymbol{v}_A + \boldsymbol{\Omega}_A \cdot (\boldsymbol{x}_k - \boldsymbol{x}_A) - \boldsymbol{v}_B - \boldsymbol{\Omega}_B \cdot (\boldsymbol{x}_k - \boldsymbol{x}_B)$, where $\boldsymbol{v}_A$ and $\boldsymbol{v}_B$ are the translational velocities of the particles, $\boldsymbol{\Omega}_A$ and $\boldsymbol{\Omega}_B$ are their spins, $\boldsymbol{x}_A$ and $\boldsymbol{x}_B$ are the radii-vectors of the centers of mass of the particles, and $\boldsymbol{x}_k$ is the radius-vector of the contact point. Rearranging the terms and neglecting the difference $\boldsymbol{\Omega}_A - \boldsymbol{\Omega}_B$ compared to the average spin $\boldsymbol{\Omega}$, we obtain the approximate relation $\boldsymbol{v}_k = \boldsymbol{v}_A - \boldsymbol{v}_B - \boldsymbol{\Omega} \cdot (\boldsymbol{x}_A - \boldsymbol{x}_B)$. The linear operator $\boldsymbol{L} \equiv \boldsymbol{v} \nabla$ is uniquely determined for three particles: $\boldsymbol{v}_A - \boldsymbol{v}_B = \boldsymbol{L} \cdot (\boldsymbol{x}_A - \boldsymbol{x}_B)$, $\boldsymbol{v}_B - \boldsymbol{v}_C = \boldsymbol{L} \cdot (\boldsymbol{x}_B - \boldsymbol{x}_C)$, and $\boldsymbol{v}_C - \boldsymbol{v}_A = \boldsymbol{L} \cdot (\boldsymbol{x}_C - \boldsymbol{x}_A)$. Using this operator, we obtain $\boldsymbol{v}_k = (\boldsymbol{L} - \boldsymbol{\Omega}) \cdot (\boldsymbol{x}_A - \boldsymbol{x}_B)$. Assuming that the vector $\boldsymbol{x}_A - \boldsymbol{x}_B$ is directed approximately along $\boldsymbol{n}_k$, we obtain the relation for the instantaneous shift along $\boldsymbol{b}_k$ that occurs owing to slipping of particles at the contact point:

$$\dot{\gamma}_k = (\boldsymbol{L}^{\mathrm{t}} + \boldsymbol{\Omega}) \circ \boldsymbol{M}_k. \tag{2}$$

For three particles, the deviatoric stress tensor $\boldsymbol{\sigma}'$ is given by

$$\boldsymbol{\sigma}' = \tau_k \boldsymbol{M}_k, \tag{3}$$

where τ_k is the shear stress acting in the $\boldsymbol{b}_k$ direction over the area with the normal $\boldsymbol{n}_k$.

We use the law of contact interaction in the form

$$\dot{\tau}_k = \alpha \dot{\gamma}_k, \tag{4}$$

where α is the tangent modulus.

To close system (2)–(4), it is necessary to determine the instantaneous angular velocity of the dyads $\boldsymbol{M}_k$, which depends mainly on the local curvature of particles at the contact point. We consider two limiting cases. For identical circular particles, the rotation of the tangent and normal vectors at the contact point is determined only by the vortex of the translational velocity field:

$$\dot{\boldsymbol{M}}_k = \boldsymbol{W} \cdot \boldsymbol{M}_k - \boldsymbol{M}_k \cdot \boldsymbol{W}, \qquad \boldsymbol{W} = (\boldsymbol{L} - \boldsymbol{L}^{\mathrm{t}})/2. \tag{5}$$

For two identical particles shaped like parallelograms, the rotation is determined only by their average angular velocity:

$$\dot{\boldsymbol{M}}_k = \boldsymbol{\Omega} \cdot \boldsymbol{M}_k - \boldsymbol{M}_k \cdot \boldsymbol{\Omega}. \tag{6}$$

Substitution of (2) and (4)–(6) into (3) yields

$$\boldsymbol{\sigma}'^{\nabla} = \alpha \boldsymbol{M}_k \boldsymbol{M}_k \circ (\boldsymbol{L}^{\mathrm{t}} + \boldsymbol{\Omega}).$$

Here the superscript ∇ denotes the corotational derivative: $\dot{\boldsymbol{\sigma}}' + \boldsymbol{\sigma}' \cdot \boldsymbol{W} - \boldsymbol{W} \cdot \boldsymbol{\sigma}'$ (Jaumann derivative) or $\dot{\boldsymbol{\sigma}}' + \boldsymbol{\sigma}' \cdot \boldsymbol{\Omega} - \boldsymbol{\Omega} \cdot \boldsymbol{\sigma}'$ (spin derivative).

The equations of state of a similar structure can be obtained from other considerations which lead to the dual definitions

$$\boldsymbol{L}^{\mathrm{t}} = \dot{\gamma}_k \boldsymbol{M}_k - \boldsymbol{\Omega}, \qquad \tau_k = \boldsymbol{\sigma} \circ \boldsymbol{M}_k$$

instead of (2) and (3). As a result, we have

$$\boldsymbol{\sigma}'^{\nabla} = \alpha (\boldsymbol{M}_k \boldsymbol{M}_k)^{-1} \circ (\boldsymbol{L}^{\mathrm{t}} + \boldsymbol{\Omega})$$

(the tensor $\boldsymbol{M}_k \boldsymbol{M}_k$ is invertible as an operator relating arbitrary deviators).

For affine deformation, the equations of state take the final form

$$\boldsymbol{\sigma}^{\nabla} = \boldsymbol{H} \circ (\nabla \boldsymbol{v} + \boldsymbol{\Omega}) - \dot{p}\boldsymbol{I}, \qquad \nabla \cdot \boldsymbol{v} = 0, \tag{7}$$

where $\boldsymbol{H}$ is the affine-stiffness tensor, $\boldsymbol{I} = \boldsymbol{e}_\alpha \boldsymbol{e}_\alpha$, and $p = -(\operatorname{tr}\boldsymbol{\sigma})/2$ is the hydrostatic pressure.

For nonaffine deformation, the structure of the equations of state can also be revealed by considering three particles. We assume that in-pair interaction between the particles is represented not only by forces but also by moment pairs that distort the field of particle rotation. The linear operator $\boldsymbol{\Gamma} \equiv \boldsymbol{\omega}\nabla$ is determined uniquely for three particles: $\boldsymbol{\omega}_A - \boldsymbol{\omega}_B = \boldsymbol{\Gamma} \cdot (\boldsymbol{x}_A - \boldsymbol{x}_B)$, $\boldsymbol{\omega}_B - \boldsymbol{\omega}_C = \boldsymbol{\Gamma} \cdot (\boldsymbol{x}_B - \boldsymbol{x}_C)$, and $\boldsymbol{\omega}_C - \boldsymbol{\omega}_A = \boldsymbol{\Gamma} \cdot (\boldsymbol{x}_C - \boldsymbol{x}_A)$. Taking this into account, we obtain

$$\dot{\theta}_k = \boldsymbol{\Gamma}^{\mathrm{t}} \circ \boldsymbol{N}_k. \tag{8}$$

Here θ_k is the curvature along the direction $\boldsymbol{n}_k$ and $\boldsymbol{N}_k \equiv \boldsymbol{n}_k \boldsymbol{e}_3$. For three particles, the tensor of moment pairs is expressed in terms of the moments m_k along the directions $\boldsymbol{n}_k$ as

$$\boldsymbol{\mu} = m_k \boldsymbol{N}_k. \tag{9}$$

Taking the law of contact interaction in the form

$$\dot{m}_k = \beta \dot{\theta}_k$$

(β is the modulus of flexural rigidity), with allowance for (8) and (9), and arbitrary rotation $\boldsymbol{N}_k$, we obtain

$$\dot{\boldsymbol{\mu}} = \beta \boldsymbol{N}_k \boldsymbol{N}_k \circ \boldsymbol{\Gamma}^{\mathrm{t}}.$$

Here, we take into account that $\dot{\boldsymbol{e}}_3 \equiv \boldsymbol{0}$ and $\boldsymbol{\omega} = \omega \boldsymbol{e}_3$, and the only nonzero components of the tensor $\boldsymbol{\mu}$ are μ_{13} and μ_{23}. Using the dual definitions

$$\boldsymbol{\Gamma}^{\mathrm{t}} = \dot{\theta}_k \boldsymbol{N}_k, \qquad m_k = \boldsymbol{\mu} \circ \boldsymbol{N}_k,$$

we obtain the equation with the same structure

$$\dot{\boldsymbol{\mu}} = \beta (\boldsymbol{N}_k \boldsymbol{N}_k)^{-1} \circ \boldsymbol{\Gamma}^{\mathrm{t}}$$

(the tensor $\boldsymbol{N}_k \boldsymbol{N}_k$ is invertible as a linear operator relating arbitrary tensors of the form $\xi \boldsymbol{e}_1 \boldsymbol{e}_3 + \eta \boldsymbol{e}_2 \boldsymbol{e}_3$). We finally assume that

$$\dot{\boldsymbol{\mu}} = \boldsymbol{D} \circ \nabla \boldsymbol{\omega}, \tag{10}$$

where $\boldsymbol{D}$ is the flexural-rigidity tensor.

Equations (1), (7), and (10) form a closed system. Below, we will distinguish models with the Jaumann and spin corotational derivatives.

Model Problem. We consider an unbounded two-dimensional medium with uniformly distributed material properties. The medium is compressed along the $\boldsymbol{e}_2$ axis by the uniform stress $\sigma_{22} \equiv -\sigma$ ($\sigma \geqslant 0$). In the configuration K, $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{t}}$, $\boldsymbol{\mu} = \boldsymbol{0}$, $\nabla\boldsymbol{\sigma} = \nabla\boldsymbol{\mu} = \boldsymbol{0}$, and $\boldsymbol{e}_\alpha$ are the eigenvectors of the tensor $\boldsymbol{\sigma}$, which possesses orthotropic symmetry with the axes directed along $\boldsymbol{e}_\alpha$, and equations of the system become linear equations with constant coefficients. The commutator of the corotational derivative $\boldsymbol{\sigma}^{\nabla}$ contains a bifurcational parameter σ.

We assume that the rigidity-modulus tensors $\boldsymbol{H}$ and $\boldsymbol{D}$ have cubic symmetry with the $\boldsymbol{e}_\alpha$ axes. In the orthogonal basis $\boldsymbol{h}_i \boldsymbol{h}_j$ [$\boldsymbol{h}_1 = (\boldsymbol{e}_1\boldsymbol{e}_1 - \boldsymbol{e}_2\boldsymbol{e}_2)/\sqrt{2}$, $\boldsymbol{h}_2 = (\boldsymbol{e}_1\boldsymbol{e}_2 + \boldsymbol{e}_2\boldsymbol{e}_1)/\sqrt{2}$, and $\boldsymbol{h}_3 = (\boldsymbol{e}_1\boldsymbol{e}_2 - \boldsymbol{e}_2\boldsymbol{e}_1)/\sqrt{2}$], the tensor $\boldsymbol{H}$ becomes

$$H_{ij} = \begin{bmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{bmatrix}$$

($\mu_1 = \mu_2$ in the particular case of isotropy). In the orthogonal basis $\boldsymbol{d}_\alpha \boldsymbol{d}_\beta$ ($\boldsymbol{d}_1 = \boldsymbol{e}_1\boldsymbol{e}_3$ and $\boldsymbol{d}_2 = \boldsymbol{e}_2\boldsymbol{e}_3$), the tensor $\boldsymbol{D}$ has the form

$$D_{ij} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

(for cubic symmetry and isotropy). We consider only one instability mechanism — geometrical softening and ignore degradation of the rigidity moduli by assuming that $\mu_i > 0$ and $\lambda > 0$.

With allowance for the above assumptions, the dynamic equations (1) take the form $\dot\sigma_{\alpha\beta,\alpha} = \rho\ddot v_\beta$ and $\dot\mu_{13,1} + \dot\mu_{23,2} + \dot\sigma_{12} - \dot\sigma_{21} = J\ddot\omega$. Combining these equations with the equations of state and incompressibility (7) and (10) and eliminating $\dot p$, for the Jaumann-derivative model, we obtain the system

$$\begin{gathered}(\mu_2+\mu_3+\sigma)\varphi_{,1111} + 2(2\mu_1-\mu_2+\mu_3)\varphi_{,1122} + (\mu_2+\mu_3-\sigma)\varphi_{,2222} + 2\mu_3\Delta\omega = \rho\Delta\ddot\varphi,\\ \bar\lambda\Delta\omega - \omega - \Delta\varphi/2 = \bar J\ddot\omega,\end{gathered} \tag{11}$$

and for the spin-derivative model, we have

$$\begin{gathered}(\mu_2+\mu_3)\varphi_{,1111} + 2(2\mu_1-\mu_2+\mu_3)\varphi_{,1122} + (\mu_2+\mu_3)\varphi_{,2222} + 2(\sigma+\mu_3)\omega_{,22} - 2(\sigma-\mu_3)\omega_{,11} = \rho\Delta\ddot\varphi,\\ \bar\lambda\Delta\omega - \omega - \Delta\varphi/2 = \bar J\ddot\omega.\end{gathered} \tag{12}$$

Here φ is the stream function of translational velocities, determined by $v_1 = \varphi_{,2}$ and $v_2 = -\varphi_{,1}$ and $\Delta \equiv \nabla\cdot\nabla$. In the second equations of systems (11) and (12), the parameters $\bar\lambda \equiv \lambda/(2\mu_3) > 0$ (whose dimension is $[\mathrm{m}^2]$) and $\bar J \equiv J/(2\mu_3) > 0$ (whose dimension is $[\mathrm{sec}^2]$) are related to the characteristic scale and time.

We seek solutions of systems (11) and (12) bounded at infinity in the general form

$$\varphi = \mathrm{Re}\,[\varphi_0 \exp(i\boldsymbol{k}\cdot\boldsymbol{x} + wt)], \qquad \omega = \mathrm{Re}\,[\omega_0 \exp(i\boldsymbol{k}\cdot\boldsymbol{x} + wt)], \qquad \boldsymbol{k} \neq \boldsymbol{0}, \tag{13}$$

where Re is the real part, φ_0 and ω_0 are the complex amplitudes, i is the imaginary unit, $\boldsymbol{k}$ is the real wave vector, and w is the complex velocity. The case $\boldsymbol{k} = \boldsymbol{0}$ is of no interest since it corresponds to the motion of the medium as a rigid body. The lines $\boldsymbol{k}\cdot\boldsymbol{x} = kC$, where $k = |\boldsymbol{k}|$ and $C = \mathrm{const}$, are the characteristics of the coordinate part of the operator for $\bar\lambda = 0$. The vector $\boldsymbol{k} = k_\alpha \boldsymbol{e}_\alpha$ is represented as $k_\alpha = k\{-\sin\delta, \cos\delta\}$, where δ is the slope of this characteristic with respect to the direction $\boldsymbol{e}_1$. By virtue of the symmetry of the problem $0 < x \leqslant 1$, we denote $\cos 2\delta \equiv x$.

Substituting (13) into (11) and (12), we obtain a homogeneous linear algebraic system for φ_0 and ω_0. The existence of a nonzero solution of this system implies the existence of nontrivial solutions of (13), which is written as

$$\rho\bar J w^4 + cw^2 + d = 0, \tag{14}$$

where

$$\begin{aligned} c &= \rho(1+\bar\lambda k^2) + \bar J[(\mu_2-\mu_1)x^2 + \mu_1 + \mu_3 - \sigma x]k^2,\\ d &= [(\mu_2-\mu_1)x^2 + \mu_1 + \mu_3 - \sigma x](1+\bar\lambda k^2)k^2 - \mu_3 k^2\end{aligned}$$

for the Jaumann-derivative model and

$$\begin{aligned} c &= \rho(1+\bar\lambda k^2) + \bar J[(\mu_2-\mu_1)x^2 + \mu_1 + \mu_3]k^2,\\ d &= [(\mu_2-\mu_1)x^2 + \mu_1 + \mu_3](1+\bar\lambda k^2)k^2 - (\sigma x + \mu_3)k^2\end{aligned}$$

for the spin-derivative model. The solutions of Eq. (14) have the form

$$w_\pm^2 = (-c \pm \sqrt{c^2 - 4\rho\bar J d})/(2\rho\bar J). \tag{15}$$

One can easily show that $c^2 - 4\rho J d > 0$ for both models. The positive coefficient d changes the sign at the points

$$\sigma = (\mu_2-\mu_1)x + [\mu_1 + \mu_3 - \mu_3/(1+\bar\lambda k^2)]/x, \tag{16}$$

or

$$\sigma = (\mu_2-\mu_1)(1+\bar\lambda k^2)x + [\mu_1 + (\mu_3+\mu_1)\bar\lambda k^2]/x \tag{17}$$

for models (11) or (12), respectively. For the first model, $c > 0$ when d passes through zero [to verify this, express σ from the equation $c = 0$ and compare the result with (16)]; for the second model, $c > 0$ for any stress.

When σ passes through the critical state, two conjugate roots $\pm\sqrt{w_+}$ move toward zero along the imaginary axis on the complex plane and, after meeting at zero, move away along the real axis; in the process, the other two conjugate imaginary roots $\pm\sqrt{w_-}$ move away from zero. This means that the divergent (static) instability mode develops directly behind the critical point [10].

The critical value of σ is determined by minimizing expressions (16) and (17) with respect to $0 < x \leqslant 1$ and $k^2 > 0$. For $\mu_2 \leqslant 2\mu_1$ (and also in the case of isotropy where $\mu_2 = \mu_1$), the horizontal characteristic $\delta_0 = 0$ appears at the moment $\sigma_0 = \mu_2$. For $\mu_2 > 2\mu_1$, a pair of characteristics that form the angles $\pm\delta_0$, where $\delta_0 = (1/2)\arccos\sqrt{\mu_1/(\mu_2-\mu_1)} \leqslant \pi/4$, appears at the moment $\sigma_0 = 2\sqrt{(\mu_2-\mu_1)\mu_1}$. The minimum of σ corresponds to $k^2 = 0$; therefore, no structures are formed at the critical moment.

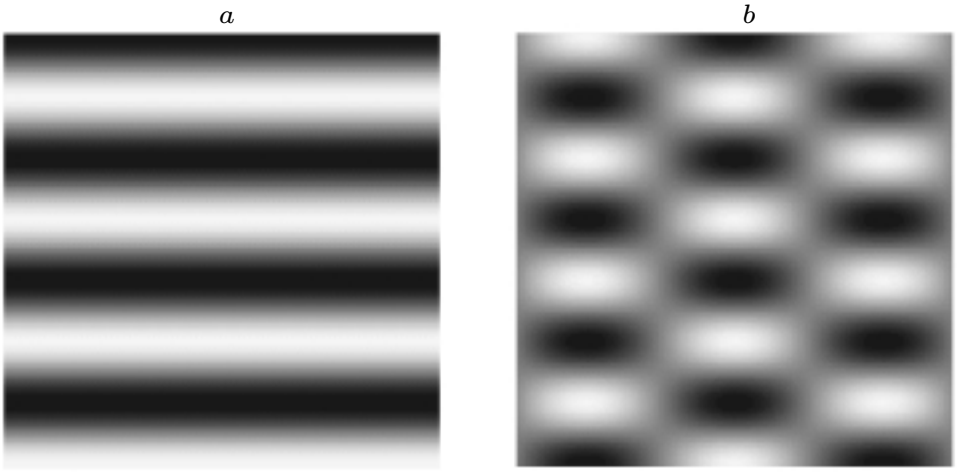

Fig. 2

If the stress exceeds the critical value, a structure is formed whose parameters x_* and k_* depend on $\sigma - \sigma_0 \equiv \Delta\sigma > 0$ and correspond to the maximum growth rate $w_+ > 0$. In the neighborhood of the critical point, it follows from (15) that $w_\pm^2 \approx -d/c$, where $|d| \ll c$; therefore, one should minimize $d(x,k)$ or $d(x,k^2)$. One can verify that the above results remain unchanged if the terms containing products of the small parameter k^4 and a power of x are ignored in the expressions for $d(x,k^2)$. As a result, for both models we obtain

$$d = [(\mu_2 - \mu_1)x^2 + \mu_1 - \sigma x]k^2 + (\mu_1 + \mu_3)\bar{\lambda}k^4. \tag{18}$$

The admissible values of x and k^2 lie in the region $d(x,k^2) < 0$ inside the band $0 < x \leqslant 1$, $k^2 > 0$. This is equivalent to the condition

$$\bar{\lambda}k^2 < -((\mu_2 - \mu_1)x^2 - \sigma x + \mu_1)/(\mu_1 + \mu_3).$$

Since $\bar{\lambda}k^2 > 0$, then $(\mu_2 - \mu_1)x^2 - \sigma x + \mu_1 < 0$. Hence, x varies within the following intervals: $x \in (x_-, 1]$ for $\mu_2 \leqslant 2\mu_1$ and $x \in (x_-, x_+)$ for $\mu_2 \geqslant 2\mu_1$. Here $x_\pm = (\sigma \pm \sqrt{\sigma^2 - 4(\mu_2 - \mu_1)\mu_1})/(2(\mu_2 - \mu_1))$. As $\sigma \to \sigma_0$, these intervals are continuously shrunk to the points $x_0 = 1$ and $x_0 = \cos 2\delta_0$. This implies that the slope of the characteristics varies slightly if the stress slightly exceeds the critical value. Therefore, in the case $\mu_2 \leqslant 2\mu_1$, we assume that the extremum of w_+ is reached at the horizontal characteristic, i.e., $x_* \approx 1$, and the wavenumber corresponds to the local extremum of (18) for $x = 1$. As a result, we obtain the admissible values

$$k_*^2 \approx (1/\bar{\lambda})\Delta\sigma/(2(\mu_2 + \mu_3)), \qquad x_* \approx 1. \tag{19}$$

If $\mu_2 > 2\mu_1$, one needs exact expressions

$$k_*^2 = (1/\bar{\lambda})x_0\Delta\sigma/(\mu_1 + \mu_3), \qquad x_* = \sigma/(2(\mu_2 - \mu_1)) \tag{20}$$

that correspond to the local extremum of (18) in the admissible domain $x_- < x_* < x_+$ and $k_*^2 > 0$. One can easily verify that $w_+^2 > 0$ for (19) and (20) .

For $\mu_2 \leqslant 2\mu_1$ and an arbitrary stress σ, a standing wave with parameters (19) is formed along the compression axis directly behind the critical point (Fig. 2a). For $\mu_2 > 2\mu_1$, the solution is a superposition of two standing waves with parameters (20), which are normal to the characteristics. In particular, cellular solutions are possible (Fig. 2b). In passing from one cell to another, the spin changes the sign; therefore, these cells can be considered as "nuclei" of subgrains. Bifurcation leads to disorientation of the crystal lattice of these cells, i.e., to the type of fragmentation we are interested in. It should be noted that there are no cellular cells in an isotropic medium ($\mu_2 = \mu_1$). Thus, the form of the solution in the model depends on anisotropy, whereas instability of the material does not.

The dependence $k^2(\Delta\sigma)$ in (20) describes the initial stage of the process. Immediately behind the bifurcation point, k^2 increases from zero, and the characteristic size k^{-1} of the fragments decreases. When it becomes comparable with the characteristic size of inhomogeneities of the material properties, it ceases to diminish, and further evolution of the structure of a fixed size occurs. The flexural rigidity $\bar{\lambda}$ has no effect on the slope of the characteristics, but it determines the scale $k^{-1} \sim \sqrt{\bar{\lambda}}$.

Equations (11) and (12) and the results obtained are valid for the compressive force $-\sigma$ directed along $\boldsymbol{e}_2$ or the tensile force σ directed along $\boldsymbol{e}_1$, since the commutator of the corotational derivative $\boldsymbol{\sigma}^\nabla$ remains unchanged

in these cases. An analysis of the differences between the critical stresses in these tests caused by local geometrical effects should not be restricted to the use of corotational derivatives, which describe the instability of a material by taking into account only rotation of its local volume. However, the above-considered models, which use corotational derivatives, adequately predict the distortion of cells caused by an increase in σ: flattening along the compression axis $\boldsymbol{e}_2$ or elongation along the extension axis $\boldsymbol{e}_1$, which follows from the second formula in (20).

Discussion of Results. We consider the left (coordinate) side of the operator in (11) and (12), which we call the equilibrium operator. Setting $\bar{\lambda} = 0$ in this operator, one can reveal the part responsible for affine deformation. We call this part the affine equilibrium operator. For a critical stress, the strictly elliptic affine equilibrium operator becomes a parabolic ($\mu_2 \leqslant 2\mu_1$) or a hyperbolic ($\mu_2 > 2\mu_1$) operator and acquires characteristics. In this case, the profile of the solution along $\boldsymbol{k}/k$ is not determined by this operator and can be arbitrary [13]. At the critical point, the definiteness of the affine equilibrium operator relative to fields (13) changes. It is positive definite (strictly elliptic) before bifurcation, degenerate (parabolic or hyperbolic) at the bifurcation point, and negative definite behind the bifurcation point. The equilibrium operator itself remains positive definite relative to fields (13). The nonaffine part of the equilibrium operator (coefficient of $\bar{\lambda}$) picks out a trivial signal from an arbitrary profile along $\boldsymbol{k}/k$ [which corresponds to $k^2 = 0$ in (13)] at the moment of bifurcation and a modulated signal ($k^2 > 0$) behind the bifurcation point.

In the model problem considered, it is essential that the coefficients of the operator in (11) and (12) do not depend on the coordinates. This implies that affine deformation of the body is completely described by affine modes and nonaffine modes can be ignored. At the critical moment, affine modes are isolated, i.e., they cannot withstand the loads applied at the current configuration K. After that, the affine modes generate forces in addition to the applied loads. As a result, by virtue of the boundary conditions, nonaffine modes in the form of harmonic waves, which take up all the loads and bend all the fields, are activated and form a structure that can be of a cellular type.

In the limiting case $\bar{\lambda} = 0$ (purely affine operator studied in [6]), arbitrary profiles $\boldsymbol{k}/k$ composed of the basis functions (13) with any $k > 0$ that move away from the basic state with different initial velocities are admissible at the critical moment. After the critical moment, the motion is accelerated, and the mode with an infinite number of halfwaves normal to the characteristics moves with maximum velocity, which corresponds to desintegration of the medium into isolated points. In the process, beginning from the critical moment, the material cannot resist the load, i.e., it becomes unstable.

The concept of internal stability of a material (with respect to affine deformation) was introduced in [8, 9, 12]. For a current configuration K of a body, this definition is written as

$$2U_0 \equiv \boldsymbol{\sigma}^{\mathrm{H}} \circ \nabla \boldsymbol{v} > 0, \tag{21}$$

where $\boldsymbol{\sigma}^{\mathrm{H}}$ is Hill's derivative of the Cauchy stress tensor, which is equal to the material derivative of the nominal-stress tensor determined in the configuration K and takes the form $\boldsymbol{\sigma}^{\mathrm{H}} \equiv \dot{\boldsymbol{\sigma}} - \boldsymbol{v}\nabla \cdot \boldsymbol{\sigma}$ for incompressible materials. It is precisely the curve of the nominal stress versus nominal strain that reaches a peak value at the moment condition (21) fails. Inequality (21) excludes bifurcations in the problems where the affine equilibrium operator with homogeneous coefficients and boundary conditions on $\boldsymbol{v}$ is used. The affine part of the operator in (11) and (12), which coincides with the affine part of the operator of the model studied in [6], has the form

$$(\mu_2 + \sigma)\varphi_{,1111} + 2(2\mu_1 - \mu_2)\varphi_{,1122} + (\mu_2 - \sigma)\varphi_{,2222} = 0.$$

This is the Euler–Lagrange equation

$$\frac{d^2}{dx_1^2}\,\frac{\partial U_0}{\partial \varphi_{,11}} + \frac{d^2}{dx_1\,dx_2}\,\frac{\partial U_0}{\partial \varphi_{,12}} + \frac{d^2}{dx_2^2}\,\frac{\partial U_0}{\partial \varphi_{,22}} = 0$$

for $2U_0 = (1/2)(\mu_2 + \sigma)(\varphi_{,11})^2 - (\mu_2 + \sigma)\varphi_{,11}\varphi_{,22} + (1/2)(\mu_2 - \sigma)(\varphi_{,22})^2 + (2\mu_1 + \sigma)(\varphi_{,12})^2$. In passing through the critical point, the positive sign of U_0 becomes negative. This means that the "second-order energy" (21) is absorbed by affine strains before reaching this point and is released behind the critical point. Thus, before reaching the critical point, there is a channel of dissipation for energy given to the body by the external load; at the critical point, the channel disappears; behind the critical point, additional energy is released. If the body is capable of nonaffine deformation, it occurs at the critical moment, and the excess energy is absorbed owing to nonaffine deformation. Thus, behind the bifurcation point, energy is transferred from affine modes to nonaffine modes.

For systems (11) and (12), the Euler–Lagrange equations are written as

$$\frac{d^2}{dx_1^2}\,\frac{\partial U}{\partial \varphi_{,11}} + \frac{d^2}{dx_1\,dx_2}\,\frac{\partial U}{\partial \varphi_{,12}} + \frac{d^2}{dx_2^2}\,\frac{\partial U}{\partial \varphi_{,22}} = 0, \qquad \frac{d}{dx_1}\,\frac{\partial U}{\partial \omega_{,1}} + \frac{d}{dx_2}\,\frac{\partial U}{\partial \omega_{,2}} - \frac{\partial U}{\partial \omega} = 0. \tag{22}$$

For (11), the velocity potential can be constructed in the form

$$2U = 2U_0 + (1/2)\mu_3(2\omega + \Delta\varphi)^2 + \lambda(\nabla\omega)^2.$$

At the bifurcation point, we have $U_0 = 0$ but $U > 0$.

For (12), however, such a function (quadratic, since the equations are linear) does not exist. To show this, it suffices to verify that the term $\sigma\omega_{,11}$ enters the first equation in (12) if only the function U contains the term $\sigma\omega\varphi_{,11}$. In this case, the second equation in (22) contains the term $\sigma\varphi_{,11}$, which is absent in the second equation in (12). The fact that the Lagrangian U does not exist implies the nonconservative nature of the above-constructed model, which refers to circulating systems studied in [10]. These systems are typical of mechanical devices with rotational degrees of freedom, which can be adequately analyzed only by the dynamic method. We used this method above to find the wave vector that possesses the highest growth rate and determines the form of the solution within the linear-perturbation method. System (12) is nonconservative by virtue of the nature of the spin of the corotational derivative of the equations of state (7). Hill [14] showed that the only corotational derivative admissible in incremental-linear relations between energy-conjugate tensors is the Jaumann derivative, which was used in model (11). However, the requirement of energy conjugation appears to be unduly stringent for the following reasons. First, it does not allow one to take into account the shape of particles in contact, which determines the form of the corotational derivative in (7). Second, the expression for the energy $\dot\tau_k\dot\gamma_k$ is invariant with respect to the system $\boldsymbol{M}_k$ for its arbitrary spin. Therefore, two limiting cases (11) and (12) have been considered.

The model with hill-like nonlinearity (deformation curve with a descending section) was considered in the present work. Bifurcation of the model was studied by linearizing the operator in the neighborhood of a single peak. It was found that, for this model, the characteristic size of the structure k^{-1} in the postcritical region depends on the value of the supercriticality $\Delta\sigma$ and, probably, it is finally determined by the structural inhomogeneity of the material and other strongly nonlinear mechanisms.

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REFERENCES

1. V. V. Rybin, *High Plastic Strains and Failure of Metals* [in Russian], Metallurgiya, Moscow (1986).
2. B. Bay, N. Hansen, D. A. Hughes, and D. Kuhlmann-Wilsdorf, "Evolution of FCC deformation structures in polyslip," *Acta Metall. Mater.*, **40**, No. 2, 205–219 (1992).
3. A. F. Revuzhenko, *Mechanics of Elastoplastic Media and Nonstandard Analysis* [in Russian], Izd. Novosib. Gos. Univ., Novosibirsk (2000).
4. E. C. Aifantis, "Pattern formation in plasticity," *Int. J. Eng. Sci.*, **33**, No. 15, 2161–2178 (1995).
5. É. L. Aéro, "Microscale strains in a two-dimensional lattice — structural transitions and bifurcation at critical shear," *Fiz. Tverd. Tela*, **42**, No. 6, 1113–1119 (2000).
6. R. Hill and J. W. Hutchinson, "Bifurcation phenomena in the plane tension test," *J. Mech. Phys. Solids*, **23**, 239–264 (1975).
7. R. J. Asaro, "Geometrical effects in the inhomogeneous deformation of ductile single crystals," *Acta Metall.*, **27**, 445–453 (1979).
8. R. Hill, "A general theory of uniqueness and stability in elastic-plastic solids," *J. Mech. Phys. Solids*, **6**, No. 3, 236–249 (1958).
9. R. Hill, "Some basic principles in the mechanics of solids without a natural time," *J. Mech. Phys. Solids*, **7**, No. 3, 209–225 (1959).
10. H. Ziegler, *Principles of Structural Stability*, Blaisdell, Toronto (1968).
11. W. Nowacki, *Theory of Elasticity* [Russian translation], Mir, Moscow (1975).
12. V. D. Klyushnikov, *Lecture Notes on the Stability of Deformable Systems* [in Russian], Izd. Mosk. Univ., Moscow (1986).
13. R. Hill, "Acceleration waves in solids," *J. Mech. Phys. Solids*, **10**, No. 1, 1–16 (1962).
14. R. Hill, "On constitutive inequalities for simple materials. Part 1," *J. Mech. Phys. Solids*, **16**, No. 4, 229–242 (1968).